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Results in Fuzzy Probabilistic Menger Cone Metric Spaces

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Abstract

A fixed point theorem in fuzzy probabilistic Menger cone metric space with implicit relation is the outcome of this paper, which lies on the axis of the cone. Also in the support of the theorem an example is given.

Keywords: Fuzzy Probabilistic Cone Metric Space, Cauchy Sequence, Fixed Point.

2010 AMS Classification

47H10, 54H25

Introduction

By using distribution functions as values of metric, Menger [2] introduced Menger space. However we consider non-negative real numbers as values of the metric in complete metric space. In the investigation of physical quantities and physiological threshold probabilistic generalization of metric spaces appears. It is also a fundamental importance in probabilistic functional analysis. Schweizer and Sklar [4] studied this concept and then the important development of Menger space theory was due to Sehgal and Bharucha-Reid [5].The development of fixed point theory in PM-spaces was due to Schweizer and Sklar [3], [4].

Review of Literature

Huang and Zhang [1] generalized the concept of metric space and defined the cone metric spaces, also proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. There are lot of works on Banach contraction principle. This principle has been extended kind of contraction mappings by [6]. Recently in 2012 Rajesh Shrivastav, Vivek Patel and Vanita Ben Dhagat [7] have given the definition of fuzzy probabilistic metric space and proved fixed point theorem. In this present paper we have considered a Complete Fuzzy Menger cone metric space under suitable condition (implicit relation) and fixed point is obtained. In support an example is also given.

Aim of the Study

In this our proposed piece of work we are trying to established a fixed point theorem in fuzzy probabilistic Menger cone metric space with implicit relation, which lies on the axis of the cone, the outcome will generalize the earlier results [6] and [7],.Also will try to produce an example in the support of the theorem.

Before going to our main result some already established definitions and lemmas are required:

Preliminary

Definition 2.1

Let (E,τ) be a topological vector space and P a subset of E, P is called a cone if,

1. P is non-empty and closed, $P \neq \{0\}$,
2. For $x,y \in P$ and $a,b \in R \Rightarrow ax + by \in P$ where $a,b \geq 0$
3. If $x \in P$ and $-x \in P \Rightarrow x = 0$

For a given cone $P \subseteq E$, a partial ordering $\geq$ with respect to P is defined by $x \geq y$ if and only if $x - y \in P$, $x > y$ if $x \geq y$ and $x \neq y$, while $x \gg y$ will stand for $x - y \in int\, P$, $int\, P$ denotes the interior of P.

Definition 2.2

A fuzzy probabilistic metric space (FPM space) is an ordered pair (X, F_α) consisting of a nonempty set X and a mapping F_α from $X \times X$ into the collections of all fuzzy distribution functions $F_\alpha \in R$ for all $\alpha \in [0,1]$. For $x,y \in X$ we denote the fuzzy distribution function $F_\alpha(x,y)$ by $F_{\alpha(x,y)}$ and $F_{\alpha(x,y)}(u)$ is the value of $F_{\alpha(x,y)}$ at u in R.

The functions $F_{\alpha(x,y)}$ for all $\alpha \in [0,1]$ assumed to satisfy the following conditions:

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1. $F_{\alpha(x,y)}\ (u) = 1\ \forall\ u > 0$ iff $x = y$,
2. $F_{\alpha(x,y)}\ (0) = 0\ \forall\ x\ ,y$ in X,
3. $F_{\alpha(x,y)} = F_{\alpha(y,x)}\ \forall\ x\ ,y$ in X,
4. If $F_{\alpha(x,y)}$ (u) = 1 and $F_{\alpha(y,z)}$ (v) = 1 $\Rightarrow F_{\alpha(x,z)}$ $(u+v) = 1\ \forall\ x\ ,y\ ,z\ \in X$ and $u, v > 0$.

Definition 2.3

A commutative, associative and non-decreasing mapping $t: [0,1] \times [0,1] \rightarrow [0,1]$ is a t-norm if and only if $t(a,1) = a\ \ \forall a \in [0,1]$, $t(0,0) = 0$ and $t(c,d) \geq t(a,b)$ for $c \geq a,\ d \geq b$.

Definition 2.4

A Fuzzy Menger space is a triplet (X, F_α, t), where (X, F_α) is a FPM-space, t is a t-norm and the generalized triangle inequality

$$F_{\alpha(x,z)}\ (u+v) \geq t\left(F_{\alpha(x,z)}(u), F_{\alpha(y,z)}\ (v)\right)$$

Holds for all x, y, z in X $u, v > 0$ and $\alpha \in [0,1]$.

Definition 2.5

Let (X, F_α, t) be a Fuzzy Menger space. If $x \in X, \varepsilon > 0$ and $\lambda \in (0,1)$, then (ε, λ) -neighborhood of x, called $U_x, (\varepsilon, \lambda)$, is defined by

$U_x(\varepsilon, \lambda) = \left\{y \in X: F_{\alpha(x,y)}(\varepsilon) > (1-\lambda)\right\}$

An (ε, λ)-topology in X is the topology induced by the family $\{U_x(\varepsilon, \lambda): x \in X, \varepsilon > 0, \alpha \in$ *[0,1]* and $\lambda \in$*(0,1)* of neighborhood.

Remark

If t is continuous, then Fuzzy Menger space (X, F_α, t) is a Housdroff space in (ε, λ)-topology.

Let (X, F_α, t) be a complete Fuzzy Menger space and $A \subset X$. Then A is called a bounded set if

$$\lim_{u\to\infty} inf\ {}^{F_{\alpha(x,y)}}_{x,y\in A}(u) = 1$$

Definition 2.6

A sequence $\{x_n\}$ in (X, F_α, t) is said to be convergent to a point x in X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that $x_n \in U_x(\varepsilon, \lambda)\ \ \forall\ n \geq N$ or equivalently $F_\alpha\,(x_n, x; \varepsilon) > 1 - \lambda$ for all $n \geq N$ and $\alpha \in [0,1]$.

Definition 2.7

A sequence $\{x_n\}$ in (X, F_α, t) is said to be cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $N = N(\varepsilon, \lambda)$ such that for all $\alpha \in [0,1]$ $F_\alpha(x_n, x_m;\ \varepsilon) > 1 - \lambda\ \forall\ n, m \geq N$.

Definition 2.8

A Fuzzy Menger space (X, F_α, t) with the continuous t-norm is said to be complete if every Cauchy sequence in X converges to a point in X for all $\alpha \in [0,1]$.

Following lemmas are selected from [4] and [8] respectively in Fuzzy Menger Space.

Lemma 2.1

Let $\{x_n\}$ be a sequence in a Fuzzy Menger space (X, F_α, t) with continuous t-norm $*$ and $t * t \geq t$. If there exists a constant $k \in (0,1)$ such that $F_{\alpha(x_n, x_{n+1})}(kt) \geq F_{\alpha(x_{n-1}, x_n)}(t)$ for all $t > 0$ and $n = 1, 2, \ldots,$

Then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.2

Let (X, F_α, t) be a Fuzzy Menger space. If there exists $k \in (0,1)$ such that

$F_{\alpha(x,y)}(kt) \geq F_{\alpha(x,y)}(t)$

For all $x, y \in X$, for all $\alpha \in [0,1]$ and $t > 0$, then $x = y$.

Definition 2.9

Let M be a nonempty set and the mapping $d: M \rightarrow X$ and $P \subset X$ be a cone, satisfies the following conditions:

2.9.1 $F_{\alpha(x,y)}(u) > 1\ \forall\ x, y \in X \Leftrightarrow x = y$

2.9.2 $F_{\alpha(x,y)}(u) = F_{\alpha(y,x)}(u)\ \forall\ x, y \in X$

2.9.3 $F_{\alpha(x,y)}(u+v) \geq t(F_{\alpha(x,z)}(u), F_{\alpha(z,y)}(v)\ \forall\ x, y \in X$.

2.9.4 For any $x, y \in X$, (x, y) is non-increasing and left continuous.

Definition 2.10

Implicit Relation

Let Φ be the class of all real-valued continuous functions $\phi: (R^+)^5 \rightarrow R^+$ non -decreasing in the first argument and satisfying the following conditions:

For $x, y \geq 0$,

$x \geq \phi(\ y, 1, y, y, y)$ or $x \geq \phi(y, y, x, y, x)$ or $x \geq \phi(1, 1, y, 1, y)$ such that $x \geq y$.

Main Result

Theorem 3.1

Let (X, F_α, t) be a complete Fuzzy Menger cone metric space and let M be a nonempty separable closed subset of cone metric space X and let T be continuous mapping defined on M satisfying contraction.

$$F_{\alpha\ T(x),T(y)}(p) \geq \phi(F_{\alpha(x,y)}(p), F_{\alpha\ y,T(x)}(p), F_{\alpha\ x,T(x)}(p), F_{\alpha\ x,T(y)}(p) \quad \ldots\ldots\ldots\ldots 3.1.1$$

for all $x, y \in X$. Then T has a fixed point in X.

Proof: For each $x \in X$ and $n \geq 1$, let $x_1 = Tx_0$ and $x_{n+1} = T\,(x_n) = T^{n+1}x_0$. Then

$F_{\alpha(x_n, x_n+1)}(p) = F_{\alpha(T(x_{n-1}), T(x_n))} \geq$

$\phi(F_{\alpha(x_{n-1}, x_n)}(p), F_{\alpha(x_n, T(x_n+1)}(p), F_{\alpha(x_n, T(x_n))}(p), F_{\alpha\left(x_{n-1}, T(x_{n-1})\right)}(p), F_{\alpha(x_n, T(x_{n-1}))}(p))$

$\geq \phi(F_{\alpha(x_{n-1}, x_n)}(p), F_{\alpha(x_n, x_n)}(p), F_{\alpha(x_n, x_{n-1})}(p), F_{\alpha(x_{n-1}, x_n)}(p), F_{\alpha(x_{n-1}, x_{n-1})}(p))$

$\phi(F_{\alpha(x_{n-1}, x_n)}(p), 1, F_{\alpha(x_n, x_{n+1})}(p), F_{\alpha(x_{n-1}, x_n)}(p), t(F_{\alpha(x_{n-1}, x_n)}(u), F_{\alpha(x_n, x_{n+1})}(v)))$

Hence from (2.10) we have

$F_{\alpha(x_n, T(x_{n-1}))}(p) \geq F_{\alpha(x_{n-1}, x_n)}(p)$

Similarly

$F_{\alpha(x_{n-1}, x_n)}(p) \geq F_{\alpha(x_{n-2}, x_{n-1})}(p)$

Hence $F_{\alpha(x_n, x_{n+1})}(p) \geq F_{\alpha(x_{n-1}, x_n)}(p) \geq F_{\alpha(x_{n-2}, x_{n-1})}(p)$

On continuing this process

$$F_{\alpha(x_n,x_{n+1})}(p) \geq F_{\alpha(x_{n-2},x_{n-1})}(p) \geq F_{\alpha(x_{n-3},x_{n-2})}(p) \geq \cdots \ldots \ldots \geq F_{\alpha(x_0,x_1)}(p)$$

Therefore the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Since (X, F_α, t) is complete, there exists $z \in X$ such that $x_n \to z$.

$F_{\alpha(z,Tz)}(p) \geq t(F_{\alpha(z,x_{n+1})}(p), F_{\alpha(x_{n+1},Tz)}(p))$

$= t(F_{\alpha(x_{n+1},Tz)}(p), F_{\alpha(Tx_n,Tz)}(p))$

$\geq t(F_{\alpha(z,x_{n+1})}(p), \emptyset\left(F_{\alpha(x_n,z)}(p), F_{\alpha(z,Tx_n,)}(p), F_{\alpha(Z,Tz)}(p), F_{\alpha(x_n,Tx_n)}(p), F_{\alpha(x_nT,z)}(p)\right))$

$\geq t(F_{\alpha(z,x_{n+1})}(p),, \emptyset\left(F_{\alpha(x_n,z)}(p), F_{\alpha(z,x_{n+1})}(p), F_{\alpha(z,Tz)}(p), F_{\alpha(x_n,x_{n+1})}(p), F_{\alpha(x_n,Tz)}(p)\right))$

Taking n→∞ we have

$F_{\alpha(z,Tz)}(p) \geq t(1, \emptyset\left(1,1, F_{\alpha(z,Tz)}(p), 1, F_{\alpha(z,Tz)}(p)\right))$

(I) If $F_{\alpha(z,Tz)}(p) \geq 1$

If $F_{\alpha(z,Tz)}(p) \in extP$. But $F_{\alpha(z,Tz)}(p) \in P$.

Therefore $F_{\alpha(z,Tz)} = 1$ and so $Tz = z$.

(II) If $F_{\alpha(z,Tz)}(p) \geq F_{\alpha(z,Tz)}(p)$

$\Longrightarrow Tz = z$.

This completes the proof.

Example

Let $M = R$ and $P = \{x \in M: x \geq 0\}$ $Let\ X = [0, \infty)$ and metric d is defined by $d_\alpha(x,y) = \frac{\alpha|x-y|}{\alpha+|x-y|}$, For each p define $F_{\alpha(x,y)}(p) = \begin{cases} 1 & for\ x = y \\ H(p) & for\ x = y \end{cases}$,

Where $H(p) = \begin{cases} 0 & if\ \ p \leq 0 \\ pd_\alpha(x,y) & if\ \ 0 < p < 1 \\ 1 & if\ \ p \geq 1 \end{cases}$

Clearly, (X, F_α, p) is a complete Fuzzy probabilistic space where t is defined by $t(p,p) \geq p$ and $\alpha \in [0,1]$.

The sequence $\{x_n\}$ is defined as $x_n = 2 - \frac{1}{2^n}$. $Tx = \begin{cases} x & 0 \leq x \leq 1 \\ \frac{4-x}{2} & x > 1 \end{cases}$,

We see the all conditions of **Theorem 3.1** are satisfied and hence 1 is the common fixed point in X.

Conclusion

So we have observed that in Fuzzy Pr**o**babilistic Menger Cone Metric Space continuity is required. With the required condition we find that fixed point lie on axis of the cone also supported by an example.

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